Tapered Dirichlet and Fejér kernels

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Contents

1 Introduction

Tapered versions of the classical Dirichlet's and Fejér's kernels are defined. The expressions as trigonometric series make some properties trivial. For some others to be obtained, coefficients capturing the taper are used. The majority of the properties provided have a descriptive character. The most important, from a theoretical point of view, are:

- Item 6 of Lemma 1
- Item 7 of Lemma 3
- Item 8 of Lemma 3

- Item 11 of Lemma 3
- Item 11 of Lemma 4
- Item 13 of Lemma 4

2 Definitions

Definition 1 *(Dahlhaus, 1997) For a complex-valued function f,*

$$H_N(f(\cdot),\lambda) := \sum_{s=0}^{N-1} f(s)e^{-i\lambda s}, \tag{1}$$

and, for the data taper $h(x)$,

$$H_{k,N}(\lambda) := H_N\left(h\left(\frac{\cdot}{N}\right)^k, \lambda\right) \tag{2}$$

and

$$H_N(\lambda) = H_{1,N}(\lambda). \tag{3}$$

Definition 2 *(Dahlhaus, 1997) For the data taper $h(x)$,*

$$H_k := \int_0^1 h(u)^k du. \tag{4}$$

Remark

$$\frac{1}{N}\sum_{r=0}^{N-1} h\left(\frac{r}{N}\right)^k = \frac{1}{N}H_{k,N}(0) \longrightarrow H_k. \tag{5}$$

3 Classical versions

The classical Dirichlet's and Fejér's kernels are, for $M \in \mathbb{N}$, respectively,

Definition 3

$$D_M(\mu) := \frac{1}{2\pi}\sum_{n=-M}^{+M} e^{-i\mu n}, \tag{6}$$

and

Definition 4

$$F_M(\mu) := \frac{1}{M}[D_0(\mu) + D_1(\mu) + \cdots + D_{M-1}(\mu)] = \frac{1}{M}\sum_{m=0}^{M-1} D_m(\mu). \tag{7}$$

Some other expressions of these kernels are

$$D_M(\mu) = \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{M} cos(\mu n)\right], \tag{8}$$

$$D_M(\mu) = \frac{1}{2\pi}\frac{sin[(M+1/2)\mu]}{sin[\mu/2]}, \tag{9}$$

and

$$F_M(\mu) = \frac{1}{2\pi}\sum_{n=-(M-1)}^{+(M-1)}\left(1 - \frac{|n|}{M}\right)e^{-i\mu n}, \tag{10}$$

$$F_M(\mu) = \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{M-1}\left(1 - \frac{n}{M}\right)cos(\mu n)\right], \tag{11}$$

$$F_M(\mu) = \frac{1}{2\pi M}\left(\frac{sin[M\mu/2]}{sin[\mu/2]}\right)^2. \tag{12}$$

4 Taper coefficients

In this section coefficients describing the taper are defined and some of its properties obtained.

4.1 Definition

It can be written

$$\begin{aligned} |H_{k,N}(\mu)|^2 &= H_{k,N}(\mu)\overline{H_{k,N}(\mu)} = \sum_{r=0}^{N-1}\sum_{s=0}^{N-1} h\left(\tfrac{r}{N}\right)^k h\left(\tfrac{s}{N}\right)^k e^{-i\mu(r-s)} \\ &= \sum_{n=-(N-1)}^{+(N-1)} e^{-i\mu n} \sum_{\{r-s=n\}} h\left(\tfrac{r}{N}\right)^k h\left(\tfrac{s}{N}\right)^k \\ &= \sum_{n=-(N-1)}^{+(N-1)} c_{n,k,N} e^{-i\mu n} \end{aligned} \tag{13}$$

where

Definition 5

$$c_{n,k,N} := \sum_{\{r-s=n\}} h\left(\tfrac{r}{N}\right)^k h\left(\tfrac{s}{N}\right)^k, \tag{14}$$

for $n = 0, +1, -1, \ldots, +(N-1), -(N-1)$.

By applying again the same idea,

$$\begin{aligned}
|H_{k,N}(\mu)|^4 &= \sum_{n_1=-(N-1)}^{+(N-1)} c_{n_1,k,N} e^{-i\mu n_1} \sum_{n_2=-(N-1)}^{+(N-1)} c_{n_2,k,N} e^{-i\mu n_2} \\
&= \sum_{n_1=-(N-1)}^{+(N-1)} \sum_{n_2=-(N-1)}^{+(N-1)} c_{n_1,k,N} c_{n_2,k,N} e^{-i\mu(n_1+n_2)} \\
&= \sum_{n=-2(N-1)}^{+2(N-1)} d_{n,k,N} e^{-i\mu n} \qquad (15)
\end{aligned}$$

where

Definition 6

$$d_{n,k,N} := \sum_{\{n_1+n_2=n\}} c_{n_1,k,N} c_{n_2,k,N}, \qquad (16)$$

for $n = 0, +1, -1, \ldots, +2(N-1), -2(N-1)$.

4.2 Properties

Lemma 1

1. *From the symmetry of the expression,*

$$c_{n,k,N} = c_{-n,k,N}$$

2. *By definition,*

$$c_{0,k,N} = H_{2k,N}(0)$$

3. *From (13) and $H_{k,N}(0) \in \mathbb{R}$,*

$$H_{k,N}(0)^2 = \sum_{n=-(N-1)}^{+(N-1)} c_{n,k,N}$$

4. *From 2,*

$$H_{k,N}(0)^2 = c_{0,k,N} + 2\sum_{n=1}^{N-1} c_{n,k,N}$$

5. *From (13) and 2,*

$$|H_{k,N}(\mu)|^2 = c_{0,k,N} + 2\sum_{n=1}^{N-1} c_{n,k,N} cos(\mu n)$$

6. *For any n,*

$$\begin{aligned}
(N-|n|)^{-1}c_{n,k,N} &= \tfrac{1}{N-|n|}\sum_{\{r-s=n\}} h\left(\tfrac{r}{N}\right)^k h\left(\tfrac{s}{N}\right)^k \\
&= \tfrac{1}{N-|n|}\sum_{r=n}^{N-1} h\left(\tfrac{r}{N}\right)^k h\left(\tfrac{r-n}{N}\right)^k \\
&= \tfrac{1}{N-|n|}\sum_{r=n}^{N-1} h\left(\tfrac{r}{N}\right)^k h\left(\tfrac{r}{N}-\tfrac{n}{N}\right)^k \qquad (17)
\end{aligned}$$

Thus, if $h(\cdot)$ *is continuous,*

$$(N-|n|)^{-1}c_{n,k,N} \longrightarrow H_{2k} \qquad (18)$$

when $N \to +\infty$.

7. *From the orthogonality of the trigonometric system,*

$$\int_{-\pi}^{+\pi} |H_{k,N}(\mu)|^2 e^{i\mu n} d\mu = 2\pi c_{n,k,N}$$

so

$$c_{n,k,N} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} |H_{k,N}(\mu)|^2 e^{i\mu n} d\mu. \qquad (19)$$

8. *The previous expression provides*

$$|c_{n,k,N}| \leq c_{0,k,N} \qquad (20)$$

9. *When there is no taper*

$$c_{n,k,N} = N - |n| \qquad (21)$$

Lemma 2

1. *From the symmetry of the expression,*

$$d_{n,k,N} = d_{-n,k,N}$$

2. *From the definition and 1 of Lemma 1,*

$$d_{0,k,N} = \sum_{n=-(N-1)}^{+(N-1)} c_{n,k,N}c_{-n,k,N} = \sum_{n=-(N-1)}^{+(N-1)} c_{n,k,N}^2.$$

3. *From (15) and* $H_{k,N}(0) \in \mathbb{R}$,

$$H_{k,N}(0)^4 = \sum_{n=-2(N-1)}^{+2(N-1)} d_{n,k,N} \qquad (22)$$

4. From the orthogonality of the trigonometric system,

$$\int_{-\pi}^{+\pi} |H_{k,N}(\mu)|^4 e^{i\mu n} d\mu = 2\pi d_{n,k,N}$$

so

$$d_{n,k,N} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} |H_{k,N}(\mu)|^4 e^{i\mu n} d\mu. \tag{23}$$

Remark. The coefficients $c_{n,k,N}$ previously defined are important to understand the behaviour of the tapered kernels that we define. It is easy to extract some qualitative information from the definitions. By hypothesis $lim_{u\to 0} h(u) = 0$ and $lim_{u\to 1} h(u) = 0$; then, for N and n big enough the terms $h\left(\frac{r}{N}\right) h\left(\frac{s}{N}\right)$ —in $\sum_{\{r-s=n\}} h\left(\frac{r}{N}\right)^k h\left(\frac{s}{N}\right)^k$ and $\sum_{\{r-s=n\}} (r-s)^l h\left(\frac{r}{N}\right)^k h\left(\frac{s}{N}\right)^k$— tend to zero. At the same time the exponent k contributes to accelerate the convergence.

5 The tapered Dirichlet kernel

5.1 Definition

For $N \in \mathbb{N}^+$ and $M = 0, +1, \ldots, +(N-1)$ the *tapered Dirichlet kernel* is defined as

Definition 7

$$L_{M,k,N}(\mu) := \frac{1}{2\pi N^{-1}\sum_{r=0}^{N-1} h\left(\frac{r}{N}\right)^{2k}} \sum_{m=-M}^{+M} (N-|m|)^{-1} \sum_{\{r-s=n\}} h\left(\tfrac{r}{N}\right)^k h\left(\tfrac{s}{N}\right)^k e^{-i\mu m} \tag{24}$$

or

$$L_{M,k,N}(\mu) := \frac{1}{2\pi} \sum_{m=-M}^{+M} \frac{(N-|m|)^{-1}\sum_{\{r-s=n\}} h\left(\frac{r}{N}\right)^k h\left(\frac{s}{N}\right)^k}{N^{-1}\sum_{r=0}^{N-1} h\left(\frac{r}{N}\right)^{2k}} e^{-i\mu m} \tag{25}$$

The name of this kernel is based on the fact that $L_{M,k,N}(\mu)$ is equal to the classical Dirichlet kernel when there is no taper, that is, for $h(x) = \mathbb{I}\{[0,1]\}$ with $\mathbb{I}\{\}$ being the characteristic function. As in this case $c_{n,k,N} = N - |n|$, then $\frac{(N-|m|)^{-1} c_{m,k,N}}{N^{-1} c_{0,k,N}} = 1$ and the dependence on N, the parameter of the taper, disappears:

$$L_{M,k,N}(\mu) = D_M(\mu) \tag{26}$$

In fact $\{L_{M,k,N}(\mu)\}_{M,N}$ can be seen as a triangular array of functions; with the formal assumption that $h(0) \neq 0$, the first element is $L_{0,k,1}(\mu) = \frac{1}{2\pi}$. The parameter N is related with the taper, while M is related with the kernel structure itself. Asymptotically it is not a strong restriction for N to be bigger than M. When there is no taper the triangular array $\{L_{M,k,N}(\mu)\}_{M,N}$ collapses into the classical sequence $\{D_M(\mu)\}_M$.

In terms of the taper coefficients, this kernel can be written as

$$L_{M,k,N}(\mu) = \frac{1}{2\pi} \sum_{m=-M}^{+M} \frac{(N-|m|)^{-1} c_{m,k,N}}{N^{-1} c_{0,k,N}} e^{-i\mu m} \tag{27}$$

or

$$L_{M,k,N}(\mu) = \frac{1}{2\pi}\left[1 + 2\sum_{m=1}^{M} \frac{(N-m)^{-1}c_{m,k,N}}{N^{-1}c_{0,k,N}} cos(\mu m)\right]. \tag{28}$$

5.2 Properties

Lemma 3

1. *$L_{M,k,N}(\mu)$ is continuous on μ*

2. *$L_{M,k,N}(\mu)$ is even, since $L_{M,k,N}(\mu) = L_{M,k,N}(-\mu)$*

3. *$L_{M,k,N}(\mu) \in \mathbb{R}$, since $L_{M,k,N}(\mu) = \overline{L_{M,k,N}(\mu)} = L_{M,k,N}(-\mu)$*

4. *$L_{M,k,N}(\mu)$ is 2π-periodic, since so the exponential function is*

5. *$lim_{\mu\to 0}L_{M,k,N}(\mu) = \frac{1}{2\pi}\sum_{m=-M}^{+M} \frac{(N-|m|)^{-1}c_{m,k,N}}{N^{-1}c_{0,k,N}} = L_{M,k,N}(0)$, since $L_{M,k,N}(\mu)$ is continuous on μ*

6. *For any M, from (18) it holds that*

$$lim_{N\to+\infty}L_{M,k,N}(0) = \frac{1}{2\pi}\sum_{m=-M}^{+M} 1 = \frac{1}{2\pi}(2M+1)$$

7. *More generally, for any M, from (27) and (18), it holds that*

$$lim_{N\to+\infty}L_{M,k,N}(\mu) = \frac{1}{2\pi}\sum_{m=-M}^{+M} e^{-i\mu m} = D_M(\mu).$$

   That is, $L_{M,k,N}(\mu)$ is asymptotically as close to D_M as desired.

8. *From the previous property,*

$$lim_{M\to+\infty}L_{M,k,N}(\mu) = lim_{M\to+\infty}D_M(\mu) = \delta(0)$$

   (notice that by definition it is necessary for N to be bigger than M)

9. *It holds that $L_{M,k,N}(\mu) \leq L_{M,k,N}(0)$ when i) $h(\cdot)$ is non-negative or non-positive, or ii) k is even. In these cases $sup_{\mu}L_{M,k,N}(\mu) = L_{M,k,N}(0)$*

10. *From 23 below,*

$$\begin{aligned}
L_{M,k,N}(\mu) &= \frac{1}{2\pi}\sum_{m=-M}^{+M}\int_{-\pi}^{+\pi} L_{M,k,N}(\gamma)e^{i\gamma m}d\gamma e^{-i\mu m} \\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} L_{M,k,N}(\gamma)\textstyle\sum_{m=-M}^{+M} e^{i(\gamma-\mu)m}d\gamma \\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} L_{M,k,N}(\gamma)2\pi D_M(\mu-\gamma)d\gamma \\
&= \int_{-\pi}^{+\pi} L_{M,k,N}(\gamma)D_M(\mu-\gamma)d\gamma \\
&= (D_M * L_{M,k,N})(\mu). \qquad (29)
\end{aligned}$$

Since in these steps the continuity or derivability of $h(\cdot)$ have not been used, as a particular case $D_M(\mu) = (D_M * D_M)(\mu)$.

11. *From 1, 6 and 18 below it seems that $L_{M,k,N}(\mu) \longrightarrow \delta(0)$ when $N \to +\infty$, where $\delta(\cdot)$ is the Dirac's delta function. To prove this important result, item 8 could be called, but a direct argument is the following:*

$$\begin{aligned}
L_{M,k,N}(\mu) &= \frac{1}{2\pi}\sum_{n=-M}^{+M}\frac{(N-|n|)^{-1}c_{n,k,N}}{N^{-1}c_{0,k,N}}e^{-i\mu n} \\
&\longrightarrow \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} e^{-i\mu n} = \delta(0)
\end{aligned}$$

when $M \to +\infty$ (notice that, by definition, N is bigger than M). From (18), $\frac{(N-|n|)^{-1}c_{n,k,N}}{N^{-1}c_{0,k,N}} \longrightarrow 1$

12. *$L_{M,k,N}(\mu)$ is derivable and*

$$\begin{aligned}
L^{l)}_{M,k,N}(\mu) &= \frac{1}{2\pi N^{-1}\sum_{r=0}^{N-1} h\left(\frac{r}{N}\right)^{2k}}\sum_{m=-M}^{+M}(N-|m|)^{-1}\sum_{\{r-s=m\}} h\left(\tfrac{r}{N}\right)^k h\left(\tfrac{s}{N}\right)^k(-im)^l e^{-i\mu m} \\
&= \frac{1}{2\pi N^{-1}\sum_{r=0}^{N-1} h\left(\frac{r}{N}\right)^{2k}}\sum_{m=-M}^{+M}(-im)^l(N-|m|)^{-1}\sum_{\{r-s=m\}} h\left(\tfrac{r}{N}\right)^k h\left(\tfrac{s}{N}\right)^k e^{-i\mu m} \\
&= \frac{1}{2\pi N^{-1}c_{0,k,N}}\sum_{m=-M}^{+M}(-im)^l(N-|m|)^{-1}c_{m,k,N}e^{-i\mu m} \\
&= \frac{1}{2\pi}\sum_{m=-M}^{+M}(-im)^l\frac{(N-|m|)^{-1}c_{m,k,N}}{N^{-1}c_{0,k,N}}e^{-i\mu m} \qquad (30)
\end{aligned}$$

$L^{l)}_{M,k,N}$ is a continuous, even and 2π-periodic function on μ. Besides, $L^{l)}_{M,k,N}(\mu) \in \mathbb{R}$ since from the symmetry of the expression $L^{l)}_{M,k,N}(\mu) = \overline{L^{l)}_{M,k,N}(\mu)}$.

On the other hand, $L_{M,k,N}(\mu)$ is an analytic function, as it is a finite sum of analytic functions

13. *As a rough bound, when i) $h(\cdot)$ is non-negative or non-positive, or ii) k is even,*

$$|L^{l)}_{M,k,N}(\mu)| \leq M^l L_{M,k,N}(0)$$

14. *For l odd, $L^{l)}_{M,k,N}(0) = 0$ from the symmetry of the expression*

15. *For l even, when i) $h(\cdot)$ is non-negative or non-positive, or ii) k is even, it holds that $L^{l)}_{M,k,N}(0) \leq 0$ if $i^l < 0$, and $L^{l)}_{M,k,N}(0) \geq 0$ if $i^l > 0$*

16. *With usual taper functions $L''_{M,k,N}(0) \neq 0$; that is, $L_{M,k,N}(\mu)$ is not lineal in $\mu = 0$ or, geometrically, the curvature of the graph is not null in $\mu = 0$. With this taper functions $L''_{M,k,N}(0) < 0$ and $L_{M,k,N}(\mu)$ has a maximum in $\mu = 0$ when i) $h(\cdot)$ is non-negative or non-positive or ii) k is even (see the previous property)*

17. *With (30) and some computes,*

$$L^{l)}_{M,k,N}(\mu) = (D_M * L^{l)}_{M,k,N})(\mu) \tag{31}$$

18. *It holds that*

$$\int_{-\pi}^{+\pi} L_{M,k,N}(\mu) d\mu = 1.$$

Expression (24) highlights which the normalization factor is.

19. *From 2 and 18,*

$$\int_{-\pi}^{0} L_{M,k,N}(\mu) d\mu = \frac{1}{2} = \int_{0}^{+\pi} L_{M,k,N}(\mu) d\mu$$

20. *(**Incomplete**) Let $\{a_N\}$ be a sequence of positive real numbers such that $a_N \to 0$; since*

$$1 = \int_{-\pi}^{+\pi} L_{M,k,N}(\mu) d\mu = \int_{|\mu| \leq a_N} L_{M,k,N}(\mu) d\mu + \int_{|\mu| > a_N} L_{M,k,N}(\mu) d\mu$$

when the second integral tends to zero, as $a_N \to 0$, for any $a > 0$ there exists N_0 such that $a_N \leq a$ when $N > N_0$, and in this case

$$\int_{|\mu| > a} L_{M,k,N}(\mu) d\mu \leq \int_{|\mu| > a_N} L_{M,k,N}(\mu) d\mu \to 0.$$

Then, to prove the asymptotic negligibility of the tails it is enough to find a sequence $\{a_N\}$ such that the first integral of the right hand tends to 1. Since odd derivatives of $L_{M,k,N}$ in $\mu = 0$ are null,

$$L_{M,k,N}(\mu) = L_{M,k,N}(0) + \frac{1}{2} L''_{M,k,N}(\mu_0)\mu^2 \quad \textit{with} \quad \mu_0 \in (0, \mu)$$

then

$$\int_{|\mu|\le a_N} L_{M,k,N}(\mu)d\mu = 2a_N L_{M,k,N}(0)+\frac{1}{2}L''_{M,k,N}(\mu_0)\int_{|\mu|\le a_N}\mu^2 d\mu$$
$$= 2a_N L_{M,k,N}(0)+\frac{1}{3}L''_{M,k,N}(\mu_0)a_N^3.$$

If $a_N = (2L_{M,k,N}(0))^{-1}$,

$$\frac{1}{3}L''_{M,k,N}(0)a_N^3 = C\frac{L''_{M,k,N}(0)}{L_{M,k,N}(0)^3}$$

so it seems it is necessary a better bound than 13 for $L''_{M,k,N}(0)$*...*

21. *As for the classical Dirichlet's kernel, the* tapered Lebesgue constants *could be defined and studied:* $\| L_{M,k,N} \|_1$

22. $L_{M,k,N}\in L_p[-\pi,+\pi]$*, although perhaps* $\| L_{M,k,N} \|_p \longrightarrow +\infty$ *with* M

23. *From (27),*

$$c_{n,k,N} = \frac{N^{-1}}{(N-|n|)^{-1}}c_{0,k,N}\int_{-\pi}^{+\pi} L_{M,k,N}(\gamma)e^{i\gamma n}d\gamma$$
$$= \left(1-\frac{|n|}{N}\right)c_{0,k,N}\int_{-\pi}^{+\pi} L_{M,k,N}(\gamma)e^{i\gamma n}d\gamma \tag{32}$$

24. *Also from (27),*

$$\int_{-\pi}^{+\pi} L_{M,k,N}(\mu)^2 d\mu = \frac{1}{2\pi}\sum_{m=-M}^{+M}\left(\frac{(N-|m|)^{-1}c_{m,k,N}}{N^{-1}c_{0,k,N}}\right)^2$$

25. *From (30),*

$$\int_{-\pi}^{+\pi} L^{l)}_{M,k,N}(\mu)d\mu = 0. \tag{33}$$

This expresses the symmetry of $L_{M,k,N}$

26. *As* $L_{M,k,N}\in L_1[-\pi,+\pi]$*, the Fourier coefficients can be considered; for* $m\in\mathbb{Z}$*,*

$$\hat{L}_{M,k,N}(m) := \frac{1}{2\pi}\int_{-\pi}^{+\pi} L_{M,k,N}(\mu)e^{-i\mu m}d\mu$$
$$= \begin{cases} \frac{1}{2\pi}\frac{(N-|m|)^{-1}c_{m,k,N}}{N^{-1}c_{0,k,N}} & \text{if} \quad m\le N-1 \\ 0 & \text{if} \quad m > N-1 \end{cases} \tag{34}$$

Then,

$$(27) \equiv L_{M,k,N}(\mu) = \sum_{m=-M}^{+M)} \hat{L}_{M,k,N}(m)e^{-i\mu m} = \sum_{m\in\mathbb{Z}} \hat{L}_{M,k,N}(m)e^{-i\mu m} \tag{35}$$

and

$$(24)\ \equiv \int_{-\pi}^{+\pi} L_{M,k,N}(\mu)^2 d\mu = 2\pi \sum_{m=-M)}^{+M} \hat{L}_{M,k,N}(m)^2 = 2\pi \sum_{m\in\mathbb{Z}} \hat{L}_{M,k,N}(m)^2 \quad (36)$$

That is, the Fourier series expression and the Parseval's equality had been found. Notice that

$$\left|\hat{L}_{M,k,N}(m)\right| \leq \frac{1}{2\pi} \int_{-\pi}^{+\pi} L_{M,k,N}(\mu) d\mu = \frac{1}{2\pi}, \quad (37)$$

so from (34)

$$|c_{m,k,N}| \leq \left(1 - \frac{|m|}{N}\right) c_{0,k,N} \qquad \forall m \in \mathbb{Z}.$$

On the other hand, the Fourier partial sums, in general, are

$$(S_M f)(\mu) := \sum_{m=-M}^{+M} \hat{f}(m) e^{-i\mu m} = \int_{-\pi}^{+\pi} f(\gamma) D_M(\mu - \gamma) d\gamma \quad (38)$$

which for the tapered Dirichlet kernel, with $M = N - 1$, provides

$$(S_M L_{M,k,N})(\mu) = \int_{-\pi}^{+\pi} L_{M,k,N}(\gamma) D_M(\mu - \gamma) d\gamma = L_{M,k,N}(\mu) \quad (39)$$

as steps (29) states
For the derivatives, directly from (35),

$$L^{l)}_{M,k,N}(\mu) = \sum_{m=-M}^{+M} (-im)^l \hat{L}_{M,k,N}(m) e^{-i\mu m} = \sum_{m\in\mathbb{Z}} (-im)^l \hat{L}_{M,k,N}(m) e^{-i\mu m} \quad (40)$$

or, from (30) and (34),

$$\begin{aligned} L^{l)}_{M,k,N}(\mu) &= \frac{1}{2\pi} \sum_{m=-M}^{+M} (-i)^l n^l \frac{(N - |m|)^{-1} c_{n,k,N}}{N^{-1} c_{0,k,N}} e^{-i\mu m} \\ &= \sum_{m=-M}^{+M} (-in)^l \hat{L}_{M,k,N}(n) e^{-i\mu m}. \end{aligned} \quad (41)$$

To see directly that $L_{M,k,N}(\mu)$ is an analytic function (there is another justification in 12) for μ close to μ_0

$$\begin{aligned} L_{M,k,N}(\mu) &= \sum_{m=-M}^{+M} \hat{L}_{M,k,N}(m) e^{-i\mu m} \\ &= \sum_{m=-M}^{+M} \hat{L}_{M,k,N}(m) e^{-i\mu_0 m} e^{-i(\mu-\mu_0)m} \\ &= \sum_{m=-M}^{+M} \hat{L}_{M,k,N}(m) e^{-i\mu_0 m} \sum_{l=0}^{\infty} \frac{(-im)^l}{l!} (\mu - \mu_0)^l \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{m=-M}^{+M} (-im)^l \hat{L}_{M,k,N}(m) e^{-i\mu_0 m} (\mu - \mu_0)^l \\ &= \sum_{l=0}^{\infty} \frac{L^{l)}_{M,k,N}(\mu_0)}{l!} (\mu - \mu_0)^l \end{aligned}$$

6 The tapered Fejér kernel

6.1 Definition

The *tapered Fejér kernel* is defined, for $N \in \mathbb{N}^{+}$, as

Definition 8

$$K_{k,N}(\mu) := \frac{1}{2\pi \sum_{r=0}^{N-1} h\left(\frac{r}{N}\right)^{2k}} \sum_{r=0}^{N-1} \sum_{s=0}^{N-1} h\left(\frac{r}{N}\right)^{k} h\left(\frac{s}{N}\right)^{k} e^{-i\mu(r-s)} \tag{42}$$

or

$$K_{k,N}(\mu) := \frac{1}{2\pi H_{2k,N}(0)} H_{k,N}(\mu)\overline{H_{k,N}(\mu)} = \frac{1}{2\pi H_{2k,N}(0)} |H_{k,N}(\mu)|^{2}. \tag{43}$$

The name of the kernel is based on the fact that $K_{k,N}(\mu)$ is equal to the classical Fejér kernel when there is no taper, that is, when $h(x) = \mathbb{I}\{[0,1]\}$ with $\mathbb{I}\{\}$ being the characteristic function. Since in the double sum $\sum_{r=0}^{N-1}\sum_{s=0}^{N-1}$ there are N terms with $r-s=0$, $N-1$ terms with $r-s=1$ and $N-1$ with $r-s=-1$, $N-2$ terms with $r-s=2$ and $N-2$ with $r-s=-2$, ..., and 1 terms with $r-s=+(N-1)$ and 1 with $r-s=-(N-1)$, it can be written

$$\sum_{r=0}^{N-1}\sum_{s=0}^{N-1} e^{-i\mu(r-s)} = \sum_{n=-(N-1)}^{+(N-1)} e^{-i\mu n} + \sum_{n=-(N-2)}^{+(N-2)} e^{-i\mu n} + \cdots + \sum_{n=-1}^{+1} e^{-i\mu n} + 1$$

$$= 2\pi D_{N-1}(\mu) + 2\pi D_{N-2}(\mu) + \cdots + 2\pi D_{1}(\mu) + 2\pi D_{0}(\mu) = 2\pi N F_{N}(\mu), \tag{44}$$

where $D_N(\cdot)$ and $F_N(\cdot)$ are, respectively, the classical Dirichlet's and Fejér's kernels. Then, in this particular case $K_{k,N}(\mu) = \frac{1}{2\pi N} 2\pi N F_N(\mu) = F_N(\mu)$.

In terms of the taper coefficients, this kernel can be written as

$$\begin{aligned} K_{k,N}(\mu) &= \frac{1}{2\pi c_{0,k,N}} \sum_{n=-(N-1)}^{+(N-1)} c_{n,k,N} e^{-i\mu n} \\ &= \frac{1}{2\pi} \sum_{n=-(N-1)}^{+(N-1)} \frac{c_{n,k,N}}{c_{0,k,N}} e^{-i\mu n} \qquad (45) \\ &= \frac{1}{2\pi} \sum_{n=-(N-1)}^{+(N-1)} \frac{(N-|n|)^{-1} c_{n,k,N}}{N^{-1} c_{0,k,N}} \left(1 - \frac{|n|}{N}\right) e^{-i\mu n} \qquad (46) \end{aligned}$$

or

$$\begin{aligned} K_{k,N}(\mu) &= \frac{1}{2\pi H_{2k,N}(0)} \left[c_{0,k,N} + 2\sum_{n=1}^{N-1} c_{n,k,N} cos(\mu n) \right] \\ &= \frac{1}{2\pi} \left[1 + 2\sum_{n=1}^{N-1} \frac{c_{n,k,N}}{c_{0,k,N}} cos(\mu n) \right] \\ &= \frac{1}{2\pi} \left[1 + 2\sum_{n=1}^{N-1} \frac{(N-n)^{-1} c_{n,k,N}}{N^{-1} c_{0,k,N}} \left(1 - \frac{n}{N}\right) cos(\mu n) \right]. \qquad (47) \end{aligned}$$

6.2 Properties

Lemma 4

1. *$K_{k,N}(\mu)$ is continuous on μ*

2. *$K_{k,N}(\mu)$ is even, since from the symmetry of its expression $K_N(\mu) = K_N(-\mu)$*

3. *$K_{k,N}(\mu) \in \mathbb{R}$ by definition. Besides, $K_{k,N}(\mu) = \overline{K_{k,N}(\mu)} = K_{k,N}(-\mu)$*

4. *$K_{k,N}(\mu) \geq 0$ by definition*

5. *$K_{k,N}(\mu)$ is 2π-periodic, since so the exponential function is*

6. *$lim_{\mu\to 0}K_{k,N}(\mu) = \frac{1}{2\pi H_{2k,N}(0)}H_{k,N}(0)\overline{H_{k,N}(0)} = \frac{H_{k,N}(0)^2}{2\pi H_{2k,N}(0)} = K_{k,N}(0)$, since $H_{k,N}(\mu)$ is continuous on μ*

7. *As $N^{-1}H_{k,N}(0) \longrightarrow H_k$ when $N \to +\infty$,*
$$K_{k,N}(0) = \frac{H_{k,N}(0)^2}{2\pi H_{2k,N}(0)} = N\frac{N^{-2}H_{k,N}(0)^2}{2\pi N^{-1}H_{2k,N}(0)} \longrightarrow +\infty$$
when $N \to +\infty$. Besides, $K_{k,N}(0) \propto N$ for large values of N.

8. *It holds that $K_{k,N}(\mu) \leq K_{k,N}(0)$ when i) $h(\cdot)$ is non-negative or non-positive, or ii) k is even, since $|H_{k,N}(\mu)| \leq |H_{k,N}(0)|$. In these cases $sup_\mu K_{k,N}(\mu) = K_{k,N}(0)$*

9. *From the Cauchy-Schwarz's inequality, when i) $h(\cdot)$ is non-negative or non-positive, or ii) k is even,*
$$K_{k,N}(\mu) = \frac{1}{2\pi H_{2k,N}(0)}\left|\sum_{r=0}^{N-1} h\left(\tfrac{r}{N}\right)^k e^{-i\mu r}\right|^2 \leq \frac{1}{2\pi H_{2k,N}(0)}NH_{2k,N}(0) = \frac{N}{2\pi}.$$
This result can also be obtained from 8 and 7, since the mentioned inequality implies —under i) or ii)— that $H_{k,N}(0)^2 \leq NH_{2k,N}(0)$. (On the other hand $H_k^2 \leq H_{2k}$ by Jensen's inequality.)

10. *From the definition and (15),*
$$\begin{aligned} K_{k,N}(\mu)^2 &= \frac{1}{[2\pi H_{2k,N}(0)]^2}\sum_{n=-2(N-1)}^{+2(N-1)} d_{n,k,N}e^{-i\mu n} \\ &= \frac{1}{(2\pi)^2}\sum_{n=-2(N-1)}^{+2(N-1)} \frac{d_{n,k,N}}{c_{0,k,N}^2}e^{-i\mu n}. \end{aligned} \tag{48}$$

11. *From (46) and (18), $K_{k,N}$ is asymptotically as close to F_N as desired. In fact, both of them tend to the Dirac's delta function.*

12. *From 7 of Lemma 1*

$$\begin{aligned} K_{k,N}(\mu) &= \frac{1}{2\pi}\sum_{n=-(N-1)}^{+(N-1)} \frac{c_{n,k,N}}{c_{0,k,N}} e^{-i\mu n} \\ &= \frac{1}{2\pi}\sum_{n=-(N-1)}^{+(N-1)} \int_{-\pi}^{+\pi} K_{k,N}(\gamma) e^{i\gamma n} d\gamma e^{-i\mu n} \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} K_{k,N}(\gamma) \textstyle\sum_{n=-(N-1)}^{+(N-1)} e^{i(\gamma-\mu)n} d\gamma \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} K_{k,N}(\gamma) 2\pi D_{N-1}(\mu-\gamma) d\gamma \\ &= \int_{-\pi}^{+\pi} K_{k,N}(\gamma) D_{N-1}(\mu-\gamma) d\gamma \\ &= (D_{N-1} * K_{k,N})(\mu). \end{aligned} \tag{49}$$

Since in these steps the continuity or derivability of $h(\cdot)$ have not been used, as a particular case $F_N(\mu) = (D_{N-1} * F_N)(\mu)$.

13. *From 1, 7 and 20 below it seems that $K_{k,N}(\mu) \longrightarrow \delta(0)$ when $N \to +\infty$, where $\delta(\cdot)$ is the Dirac's delta function. To prove this important result, item 11 could be called, but a direct argument is the following:*

$$\begin{aligned} K_{k,N}(\mu) &= \frac{1}{2\pi}\sum_{n=-(N-1)}^{+(N-1)} \frac{(N-|n|)^{-1} c_{n,k,N}}{N^{-1} c_{0,k,N}} \left(1-\frac{|n|}{N}\right) e^{-i\mu n} \\ &\longrightarrow \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} e^{-i\mu n} = \delta(0) \end{aligned}$$

when $N \to +\infty$. On the one hand, $\frac{(N-|n|)^{-1} c_{n,k,N}}{N^{-1} c_{0,k,N}} \longrightarrow 1$ (see 6 of Lemma 1); on the other hand, $1-|n|/N \longrightarrow 1$ for any n (a similar argument is used, for example, by Priestley [1981] in Theorem 4.8.1 and in section 5.3.2.)

14. *$K_{k,N}(\mu)$ is derivable and*

$$\begin{aligned} K_{k,N}^{l)}(\mu) &= \frac{1}{2\pi H_{2k,N}(0)} \sum_{r=0}^{N-1}\sum_{s=0}^{N-1} [-i(r-s)]^l h\left(\tfrac{r}{N}\right)^k h\left(\tfrac{s}{N}\right)^k e^{-i\mu(r-s)} \\ &= \frac{1}{2\pi}\sum_{n=-(N-1)}^{+(N-1)} (-i)^l n^l \frac{c_{n,k,N}}{c_{0,k,N}} e^{-i\mu n}. \end{aligned} \tag{50}$$

$K^{l)}_{k,N}$ *is a continuous, even and* 2π*-periodic function on* μ*. Besides,* $K^{l)}_{k,N}(\mu) \in \mathbb{R}$ *since from the symmetry of the expression* $K^{l)}_{k,N}(\mu) = \overline{K^{l)}_{k,N}(\mu)}$*.*
On the other hand, $K_{k,N}(\mu)$ *is an analytic function, as it is a finite sum of analytic functions*

15. *As a rough bound, when i)* $h(\cdot)$ *is non-negative or non-positive, or ii)* k *is even,*

$$|K^{l)}_{k,N}(\mu)| \leq (N-1)^l K_{k,N}(0) < N^l K_{k,N}(0)$$

16. *For* l *odd,* $K^{l)}_{k,N}(0) = 0$ *from the symmetry of the expression*

17. *For* l *even, when i)* $h(\cdot)$ *is non-negative or non-positive, or ii)* k *is even, it holds that* $K^{l)}_{k,N}(0) \leq 0$ *if* $i^l < 0$*, and* $K^{l)}_{k,N}(0) \geq 0$ *if* $i^l > 0$

18. *With usual taper functions* $K''_{k,N}(0) \neq 0$*; that is,* $K_{k,N}(\mu)$ *is not lineal in* $\mu = 0$ *or, geometrically, the curvature of the graph is not null in* $\mu = 0$*. With this taper functions* $K''_{k,N}(0) < 0$ *and* $K_{k,N}(\mu)$ *has a maximum in* $\mu = 0$ *when i)* $h(\cdot)$ *is non-negative or non-positive or ii)* k *is even (see the previous property)*

19. *With (50) and some computes,*

$$K^{l)}_{k,N}(\mu) = (D_{N-1} * K^{l)}_{k,N})(\mu) \tag{51}$$

20. *It holds that*

$$\begin{aligned}
\int_{-\pi}^{+\pi} K_{k,N}(\mu)d\mu &= \frac{1}{2\pi H_{2k,N}(0)} \sum_{r=0}^{N-1}\sum_{s=0}^{N-1} h\left(\tfrac{r}{N}\right)^k h\left(\tfrac{s}{N}\right)^k \int_{-\pi}^{+\pi} e^{-i\mu(r-s)}d\mu \\
&= \frac{1}{2\pi H_{2k,N}(0)} \sum_{r=0}^{N-1} h\left(\tfrac{r}{N}\right)^{2k} 2\pi = 1
\end{aligned}$$

Expression (42) highlights which the normalization factor is.

21. *From 2 and 20,*

$$\int_{-\pi}^{0} K_{k,N}(\mu)d\mu = \frac{1}{2} = \int_{0}^{+\pi} K_{k,N}(\mu)d\mu$$

22. *(**Incomplete**) Let* $\{a_N\}$ *be a sequence of positive real numbers such that* $a_N \to 0$*; since*

$$1 = \int_{-\pi}^{+\pi} K_{k,N}(\mu)d\mu = \int_{|\mu|\leq a_N} K_{k,N}(\mu)d\mu + \int_{|\mu|>a_N} K_{k,N}(\mu)d\mu$$

when the second integral tends to zero, as $a_N \to 0$*, for any* $a > 0$ *there exists* N_0 *such that* $a_N \leq a$ *when* $N > N_0$*, and in this case*

$$\int_{|\mu|>a} K_{k,N}(\mu)d\mu \leq \int_{|\mu|>a_N} K_{k,N}(\mu)d\mu \to 0.$$

Then, to prove the asymptotic negligibility of the tails it is enough to find a sequence $\{a_N\}$ such that the first integral of the right hand tends to 1. Since odd derivatives of $K_{k,N}$ in $\mu = 0$ are null,

$$K_{k,N}(\mu) = K_{k,N}(0) + \frac{1}{2}K''_{k,N}(\mu_0)\mu^2 \quad \textit{with} \quad \mu_0 \in (0, \mu)$$

then

$$\begin{aligned}\int_{|\mu|\le a_N} K_{k,N}(\mu)d\mu &= 2a_N K_{k,N}(0) + \frac{1}{2}K''_{k,N}(\mu_0)\int_{|\mu|\le a_N}\mu^2 d\mu \\ &= 2a_N K_{k,N}(0) + \frac{1}{3}K''_{k,N}(\mu_0)a_N^3.\end{aligned}$$

If $a_N = (2K_{k,N}(0))^{-1}$,

$$\frac{1}{3}K''_{k,N}(0)a_N^3 = C\frac{K''_{k,N}(0)}{K_{k,N}(0)^3}$$

so it seems it is necessary a better bound than 15 for $K''_{k,N}(0)$...

23. $K_{k,N} \in L_p[-\pi, +\pi]$, *although perhaps* $\| K_{k,N} \|_p \longrightarrow +\infty$ *with* N

24. *From (45), or directly from 7 of Lemma 1,*

$$c_{n,k,N} = c_{0,k,N}\int_{-\pi}^{+\pi} K_{k,N}(\gamma)e^{i\gamma n}d\gamma \tag{52}$$

25. *From (48),*

$$\begin{aligned}\int_{-\pi}^{+\pi} K_{k,N}(\mu)^2 d\mu &= \frac{1}{(2\pi)^2}\int_{-\pi}^{+\pi}\sum_{n=-2(N-1)}^{+2(N-1)}\frac{d_{n,k,N}}{c_{0,k,N}^2}e^{-i\mu n}d\mu \\ &= \frac{1}{(2\pi)^2}\frac{d_{0,k,N}}{c_{0,k,N}^2}2\pi \\ &= \frac{1}{2\pi}\sum_{n=-(N-1)}^{+(N-1)}\left(\frac{c_{n,k,N}}{c_{0,k,N}}\right)^2.\end{aligned} \tag{53}$$

This result can also be obtained directly from (45)

26. *From (50),*

$$\int_{-\pi}^{+\pi} K_{k,N}^{l)}(\mu)d\mu = 0. \tag{54}$$

This expresses the symmetry of $K_{k,N}$.

27. ***(Incomplete)** It holds that, for j and l fixed,*

$$\begin{aligned}\int_{-\pi}^{+\pi} K_{k,N}(\mu)e^{i\mu(t_j-t_l)}d\mu &= \frac{1}{2\pi c_{0,k,N}}\int_{-\pi}^{+\pi}\textstyle\sum_{n=-(N-1)}^{+(N-1)} c_{n,k,N}e^{-i\mu n}e^{i\mu(t_j-t_l)}d\mu \\ &= \frac{1}{2\pi c_{0,k,N}}2\pi c_{t_j-t_l,k,N} = \frac{c_{t_j-t_l,k,N}}{c_{0,k,N}}\mathbb{I}\{|t_j-t_l| < N\},\end{aligned}$$

where $\mathbb{I}\{\}$ *is the characteristic function. That is, only close points provide a value different from zero for the previous integral. In this item the notation* t_j*,* t_l*,* S*,* N*... is the used in Casado (????).*
Now, for j *fixed it can be defined the influence interval of point* t_j *as*

$$I_j := \{t_l \ / \ |t_j - t_l| = |S(j-l)| < N\} \tag{55}$$

and it holds that

$$I_j = \begin{cases} \{t_l \ / \ |N(j-l)| < N\} = \{t_l \ / \ t_l = t_j\} & \text{if} \quad S = N \\ \\ \{t_l \ / \ |S(j-l)| < N\} = \{t_l \ / \ |j-l| < N/S\} & \text{if} \quad S/N \to 0 \end{cases}$$

In I_j *there are only one point when* $S = N$ *and* $2N/S$ *points when* $S/N \to 0$ *(notice that in this case* $N/S \to \infty$*). There is a subtle difference between* I_j *defined in (55) and* B_j *defined in Casado (????): the length is given by definition for the former and imposed by orthogonality properties of the trigonometric functions in the latter. Now, the sum on* l *is*

$$\sum_{l=1}^{M} \int_{-\pi}^{+\pi} K_{k,N}(\mu) e^{i\mu(t_j - t_l)} d\mu = \sum_{l=1}^{M} \frac{c_{t_j - t_l,k,N}}{c_{0,k,N}} \mathbb{I}\{t_l \in I_j\}.$$

$$= \begin{cases} \frac{c_{0,k,N}}{c_{0,k,N}} = 1 & \text{if} \quad S = N \\ \\ \frac{\sum_{l=1}^{M} c_{t_j - t_l,k,N} \mathbb{I}\{t_l \in I_j\}}{c_{0,k,N}} & \text{if} \quad S/N \to 0 \end{cases} \tag{56}$$

$$\longrightarrow \begin{cases} 1 & \text{if} \quad S = N \\ \\ ... & \text{if} \quad S/N \to 0 \end{cases} \tag{57}$$

28. *As* $K_{k,N} \in L_1[-\pi, +\pi]$*, the Fourier coefficients can be considered; for* $m \in \mathbb{Z}$*,*

$$\hat{K}_{k,N}(m) := \frac{1}{2\pi} \int_{-\pi}^{+\pi} K_{k,N}(\mu) e^{-i\mu m} d\mu = \begin{cases} \frac{1}{2\pi} \frac{c_{m,k,N}}{c_{0,k,N}} & \text{if} \quad m \le N-1 \\ \\ 0 & \text{if} \quad m > N-1 \end{cases} \tag{58}$$

Then,

$$(45) \ \equiv \ K_{k,N}(\mu) = \sum_{m=-(N-1)}^{+(N-1)} \hat{K}_{k,N}(m) e^{-i\mu m} = \sum_{m \in \mathbb{Z}} \hat{K}_{k,N}(m) e^{-i\mu m} \tag{59}$$

and

$$(53) \ \equiv \ \int_{-\pi}^{+\pi} K_{k,N}(\mu)^2 d\mu = 2\pi \sum_{m=-(N-1)}^{+(N-1)} \hat{K}_{k,N}(m)^2 = 2\pi \sum_{m \in \mathbb{Z}} \hat{K}_{k,N}(m)^2 \tag{60}$$

That is, the Fourier series expression and the Parseval's equality had been found. Notice that

$$\left| \hat{K}_{k,N}(m) \right| \le \frac{1}{2\pi} \int_{-\pi}^{+\pi} K_{k,N}(\mu) d\mu = \frac{1}{2\pi}, \tag{61}$$

so from (58)

$$|c_{m,k,N}| \leq c_{0,k,N} \qquad \forall m \in \mathbb{Z}$$

This result was also obtained directy from (19). On the other hand, the Fourier partial sums, in general, are

$$(S_M f)(\mu) := \sum_{m=-M}^{+M} \hat{f}(m) e^{-i\mu m} = \int_{-\pi}^{+\pi} f(\gamma) D_M(\mu - \gamma) d\gamma \tag{62}$$

which for the tapered Fejér kernel, with $M = N - 1$, provides

$$(S_{N-1} K_{k,N})(\mu) = \int_{-\pi}^{+\pi} K_{k,N}(\gamma) D_{N-1}(\mu - \gamma) d\gamma = K_{k,N}(\mu) \tag{63}$$

as steps (49) states.
For the derivatives, directly from (59),

$$K_{k,N}^{l)}(\mu) = \sum_{m=-(N-1)}^{+(N-1)} (-im)^l \hat{K}_{k,N}(m) e^{-i\mu m} = \sum_{m\in\mathbb{Z}} (-im)^l \hat{K}_{k,N}(m) e^{-i\mu m} \tag{64}$$

or, from (50) and (58),

$$\begin{aligned} K_{k,N}^{l)}(\mu) &= \frac{1}{2\pi} \sum_{n=-(N-1)}^{+(N-1)} \frac{(-i)^l n^l c_{n,k,N}}{c_{0,k,N}} e^{-i\mu n} \\ &= \sum_{n=-(N-1)}^{+(N-1)} (-in)^l \hat{K}_{k,N}(n) e^{-i\mu n}. \end{aligned} \tag{65}$$

To see directly that $K_{k,N}(\mu)$ is an analytic function (there is another justification in 14), for μ close to μ_0

$$\begin{aligned} K_{k,N}(\mu) &= \sum_{m=-(N-1)}^{+(N-1)} \hat{K}_{k,N}(m) e^{-i\mu m} \\ &= \sum_{m=-(N-1)}^{+(N-1)} \hat{K}_{k,N}(m) e^{-i\mu_0 m} e^{-i(\mu-\mu_0)m} \\ &= \sum_{m=-(N-1)}^{+(N-1)} \hat{K}_{k,N}(m) e^{-i\mu_0 m} \sum_{l=0}^{\infty} \frac{(-im)^l}{l!} (\mu - \mu_0)^l \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{m=-(N-1)}^{+(N-1)} (-im)^l \hat{K}_{k,N}(m) e^{-i\mu_0 m} (\mu - \mu_0)^l \\ &= \sum_{l=0}^{\infty} \frac{K_{k,N}^{l)}(\mu_0)}{l!} (\mu - \mu_0)^l \end{aligned}$$

7 Relation between the kernels

Since $\{L_{M,k,N}(\mu)\}_{M,N}$ is a triangular array of functions, sums in several directions can be studied. The results of these sums have been represented in Figure (1).

An important general relation between the tapered Dirichlet and Fejér kernels is that the latter can be expressed as a Cesàro-like sum from the former. For the N-th row, with the same idea used to obtain (44), or using that for a general sequence of numbers $\sum_{i=0}^{N-1}\sum_{j=-i}^{+i} a_j = \sum_{j=-(N-1)}^{+(N-1)}(N-|j|)a_j$, it can be written

$$\begin{aligned}
\frac{1}{N}\sum_{m=0}^{N-1} L_{m,k,N}(\mu) &= \frac{1}{N}\sum_{M=0}^{N-1}\frac{1}{2\pi}\sum_{m=-M}^{+M}\frac{(N-|m|)^{-1}c_{m,k,N}}{N^{-1}c_{0,k,N}}e^{-i\mu m} \\
&= \frac{1}{2\pi}\sum_{M=0}^{N-1}\sum_{m=-M}^{+M}\frac{(N-|m|)^{-1}c_{m,k,N}}{c_{0,k,N}}e^{-i\mu m} \\
&= \frac{1}{2\pi}\sum_{n=-(N-1)}^{+(N-1)}(N-|n|)\frac{(N-|n|)^{-1}c_{n,k,N}}{c_{0,k,N}}e^{-i\mu n} \\
&= \frac{1}{2\pi}\sum_{n=-(N-1)}^{+(N-1)}\frac{c_{n,k,N}}{c_{0,k,N}}e^{-i\mu n} \\
&= K_{k,N}(\mu) \qquad (66)
\end{aligned}$$

The previous average is not the classical Cesàro sumation, as the terms $L_{M,k,N}(\mu)$ depends on N. The sum on M explains why the kernel $K_{k,N}$ does not depend explicitly on M, only on the taper parameter N; nevertheless, there is an implicit relation between M and N in the definition of the triangular array. By using the *Fejér sums* notation, the previous result can be expressed as

$$\sigma_{N-1}L_{M,k,N} = K_{k,N}. \qquad (67)$$

The sum also shows that the first element of the sequence $\{K_{k,N}\}_N$ is $K_{k,1} = \frac{1}{2\pi}$ (with the formal assumption that $h(0) \neq 0$, as it was done for $L_{M,k,N}$). Finally, results (66) and 11 of Lemma 3 are a new proof of 13 of Lemma 4.

On the other direction, if the sum is applied to the l-th column,

$$\begin{aligned}
\frac{1}{N+1-l}\sum_{n=l}^{N} L_{M,k,n}(\mu) &= \frac{1}{N+1-l}\sum_{n=l}^{N}\frac{1}{2\pi}\sum_{m=-M}^{+M}\frac{(n-|m|)^{-1}c_{m,k,n}}{n^{-1}c_{0,k,n}}e^{-i\mu m} \\
&= \frac{1}{2\pi}\sum_{m=-M}^{+M}\frac{1}{N+1-l}\sum_{n=l}^{N}\frac{(n-|m|)^{-1}c_{m,k,n}}{n^{-1}c_{0,k,n}}e^{-i\mu m} \\
&\longrightarrow \frac{1}{2\pi}\sum_{m=-M}^{+M}e^{-i\mu m} = D_M(\mu) \qquad (68)
\end{aligned}$$

Notice that, for the coefficients,

$$\frac{1}{N+1-l}\sum_{n=l}^{N}\frac{(n-|m|)^{-1}c_{m,k,n}}{n^{-1}c_{0,k,n}} \longrightarrow 1$$

$$
\begin{array}{lllllllll}
\mathbf{N} & \mathbf{M} & \boldsymbol{L_{M,k,N}} & & & & & & \boldsymbol{K_{k,N}} \\
1 & 0 & L_{0,k,1} & & & & & = & K_{k,1} \\
2 & 0\ 1 & L_{0,k,2} & L_{1,k,2} & & & & = & K_{k,2} \\
3 & 0\ 1\ 2 & L_{0,k,3} & L_{1,k,3} & L_{2,k,3} & & & = & K_{k,3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & & \vdots \\
\text{N} & 0\ 1\ \ 2 \ldots \text{N-1} & L_{0,k,N} & L_{1,k,N} & L_{2,k,N} & \cdots & L_{N-1,k,N} & = & K_{k,N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & & \vdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\
\infty & \mathbb{N} & D_0 & D_1 & D_2 & & D_{N-1} & \longrightarrow & \boxed{\delta}
\end{array}
$$

$$
\downarrow
$$

$$
h(x) \equiv \boldsymbol{I}_{[0,1]}(x) \quad \text{(No taper)}
$$

$$
\downarrow
$$

$$
\begin{array}{lllllllll}
\mathbf{N} & \mathbf{M} & \boldsymbol{D_M} & & & & & & \boldsymbol{F_N} \\
1 & 0 & D_0 & & & & & = & F_1 \\
2 & 0\ 1 & D_0 & D_1 & & & & = & F_2 \\
3 & 0\ 1\ 2 & D_0 & D_1 & D_2 & & & = & F_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & & \vdots \\
\text{N} & 0\ 1\ \ 2 \ldots \text{N-1} & D_0 & D_1 & D_2 & \cdots & D_{N-1} & = & F_N \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & & \vdots \\
\downarrow & \downarrow & \| & \| & \| & & \| & & \downarrow \\
\infty & \mathbb{N} & D_0 & D_1 & D_2 & & D_{N-1} & \longrightarrow & \boxed{\delta}
\end{array}
$$

Figure 1: Triangular arrays of kernels, with and without taper, and the averages (or its limits) in the margin

since, from (18), $\frac{(n-|m|)^{-1}c_{m,k,n}}{n^{-1}c_{0,k,n}} \longrightarrow 1$. Another way to justify this is to consider the item 7 of Lemma 3 and the properties of the Cesàro sumation. As expected, when there is no taper these coefficients take the value 1 and the limit becomes an equality.

Finally, only the diagonal of the triangular array can be taken into account (this can be seen as a sort of synchronization of the parameters). Thus, for the sequence $\{L_{M,k,M+1}(\mu)\}_M = \{L_{N-1,k,N}(\mu)\}_N$, from (18),

$$
L_{N-1,k,N}(\mu) = \frac{1}{2\pi} \sum_{m=-(N-1)}^{+(N-1)} \frac{(N-|m|)^{-1}c_{m,k,N}}{N^{-1}c_{0,k,N}} e^{-i\mu m} \longrightarrow \sum_{m=-\infty}^{+\infty} e^{-i\mu m} = \delta. \tag{69}
$$

A different kind of relation can be obtained from the integral expressions (32) and (52); for any n

$$
\left(1 - \frac{|n|}{N}\right) \int_{-\pi}^{+\pi} L_{M,k,N}(\gamma) e^{i\gamma n} d\gamma = \int_{-\pi}^{+\pi} K_{k,N}(\gamma) e^{i\gamma n} d\gamma \tag{70}
$$

8 Summary of expressions

At this point it is interested to summarize some expressions:

Classical Dirichlet's kernel

$$D_M(\mu) := \frac{1}{2\pi} \sum_{n=-M}^{+M} e^{-i\mu n} \qquad [6]$$

$$D_M(\mu) = \frac{1}{2\pi} \left[1 + 2 \sum_{n=1}^{M} cos(\mu n) \right] \qquad [8]$$

Classical Fejér's kernel

$$F_M(\mu) := \frac{1}{M} \sum_{m=0}^{M-1} D_m(\mu) \qquad [7]$$

$$F_M(\mu) = \frac{1}{2\pi} \sum_{n=-(M-1)}^{+(M-1)} \left(1 - \frac{|n|}{M} \right) e^{-i\mu n} \qquad [10]$$

$$F_M(\mu) = \frac{1}{2\pi} \left[1 + 2 \sum_{n=1}^{M-1} \left(1 - \frac{n}{M} \right) cos(\mu n) \right] \qquad [11]$$

Tapered Dirichlet kernel

$$L_{M,k,N}(\mu) = \frac{1}{2\pi} \sum_{n=-M}^{+M} \frac{(N-|n|)^{-1} c_{n,k,N}}{N^{-1} c_{0,k,N}} e^{-i\mu n} \qquad [27]$$

$$L_{M,k,N}(\mu) = \frac{1}{2\pi} \left[1 + 2 \sum_{n=1}^{M} \frac{(N-n)^{-1} c_{n,k,N}}{N^{-1} c_{0,k,N}} cos(\mu n) \right] \qquad [28]$$

Tapered Fejér kernel

$$K_{k,N}(\mu) = \frac{1}{N} \sum_{m=0}^{N-1} L_{m,k,N}(\mu) \qquad [66]$$

$$K_{k,N}(\mu) = \frac{1}{2\pi} \sum_{n=-(N-1)}^{+(N-1)} \frac{(N-|n|)^{-1} c_{n,k,N}}{N^{-1} c_{0,k,N}} \left(1 - \frac{|n|}{N} \right) e^{-i\mu n} \qquad [46]$$

$$K_{k,N}(\mu) = \frac{1}{2\pi} \left[1 + 2 \sum_{n=1}^{N-1} \frac{(N-n)^{-1} c_{n,k,N}}{N^{-1} c_{0,k,N}} \left(1 - \frac{n}{N} \right) cos(\mu n) \right] \qquad [47]$$

That is, the tapered kernels can be expressed in the same type of sums and fulfill the similar relations than the classical ones:

- Compare expressions (27) and (28) with (6) and (8).
- Compare also expressions (46) and (47) with (10) and (11), respectively.
- Compare expression (66) with (7).
- Finally, compare the relation between expressions (27) and (28) with (46) and (47), respectively, with that between expressions (6) and (8) with (10) and (11).

9 Naïve applications

Both the tapered Dirichlet and Fejér kernels are symmetrical and tend to the Dirac's del function when $M \to \infty$. For such kernels

1. The following point convergence holds when $M \to +\infty$

$$f(\mu)S_M(\mu) \longrightarrow \begin{cases} 0 & \text{when} \quad \mu \neq 0 \\ 0 & \text{when} \quad \mu = 0 \quad \text{and} \quad f(\mu) = 0 \\ \infty & \text{when} \quad \mu = 0 \quad \text{and} \quad f(\mu) \neq 0 \end{cases}$$

2. The following weak (or in distribution) convergence holds when $M \to +\infty$

$$\textstyle\int_a^b S_M(\mu)d\mu \longrightarrow \begin{cases} 1 & \text{if} \quad a < 0 < b \\ \frac{1}{2} & \text{if} \quad a = 0 \quad \text{or} \quad b = 0 \\ 0 & \text{if} \quad b < 0 \quad \text{or} \quad a > 0 \end{cases}$$

3. The following weak (or in distribution) convergence holds when $M \to +\infty$

$$\textstyle\int_a^b f(\mu)S_M(\mu)d\mu \longrightarrow \begin{cases} f(0) & \text{if} \quad a < 0 < b \\ \frac{1}{2}f(0) & \text{if} \quad a = 0 \quad \text{or} \quad b = 0 \\ 0 & \text{if} \quad b < 0 \quad \text{or} \quad a > 0 \end{cases}$$

4. Since

$$\int_a^b f(\mu)S_M(\mu)d\mu = \int_{-\pi}^{+\pi} f(\mu)S_M(\mu)d\mu - \int_{-\pi}^{a} f(\mu)S_M(\mu)d\mu - \int_{b}^{+\pi} f(\mu)S_M(\mu)d\mu,$$

the asymptotic weak (or in distribution) convergence, when $M \to +\infty$, of this relation can be described symbolically as

$$\begin{cases} f(0) &= f(0) - 0 - 0 & \text{if} \quad a < 0 < b \\ \frac{1}{2}f(0) &= f(0) - \frac{1}{2}f(0) - 0 & \text{if} \quad a = 0 \\ \frac{1}{2}f(0) &= f(0) - 0 - \frac{1}{2}f(0) & \text{if} \quad b = 0 \\ 0 &= f(0) - 0 - f(0) & \text{if} \quad b < 0 \\ 0 &= f(0) - f(0) - 0 & \text{if} \quad a > 0 \end{cases}$$

5. If $\| f \|_\infty$ exists

$$\begin{aligned}
\left| \int_a^b f(\mu) S_M(\mu) d\mu \right| &\leq \int_a^b |f(\mu)| S_M(\mu) d\mu \\
&\leq \| f(\mu) \|_\infty \int_a^b S_M(\mu) d\mu \\
&\longrightarrow \begin{cases} \| f(\mu) \|_\infty & \text{if} \quad a < 0 < b \\ \frac{1}{2} \| f(\mu) \|_\infty & \text{if} \quad a = 0 \quad \text{or} \quad b = 0 \\ 0 & \text{if} \quad b < 0 \quad \text{or} \quad a > 0 \end{cases}
\end{aligned}$$

   when $M \to +\infty$.

6. The previous tapered kernels can allow for some existing proofs to be improved in the taper framework. This could provide slightly —but crucial— better rates of convergence. In Dahlhaus (1997) one finds an example: *We conjecture that the rate $O(N^{-2})$ cannot be improved with a periodogram type estimator. A periodogram without taper would lead to a bias of $O(N^{-1})$ and therefore to $\sqrt{N}/N \to 0$ which contradicts $N/\sqrt{N} \to 0$. Thus, without taper it is not possible to achieve $\sqrt{T}$-consistency at all. It is noteworthy that the use of a data taper does not lead to an increase of the variance if $S/N \to 0$.* The improvement of such existing proofs would be based on the idea of introducing the taper in the process and changing asymptotically, in the proof, the tapered versions of the kernels by the classical ones wherever the former appear.

7. Another application takes place in the proof that has motivated these definitions (in Casado, ????)...

8. ...

References

[1] Casado, D. (????). Asymptotic estimation for the non-stationary integrated spectrum

[2] Dahlhaus, R. (1997). Fitting time series models to nonstationary processes *The Annals of Statistics*. 25 (1), 1–37.

[3] Priestley, M.B. (1981). *Spectral Analysis and Time Series*. Elsevier (Eleventh printing 2001. Reprinted 2004).